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Subject: The Astrometric Information Rate with a Central Obscuration in a Rectangular Aperture

I. Introduction.

We consider a diffraction-limited astrometric instrument with a rectangular aperture as shown in the preliminary design for the proposed FAME satellite. The astrometric measurement is to be made in a single direction (in the spacecraft frame.) For a primary of fixed mass (approximately fixed area) it is astrometrically advantageous to elongate the mirror in the direction of the measurement. The analysis below assumes the optical instrument is either a telescope or a Fizeau interferometer, but not a Michelson interferometer. That analysis is done in the bright-star limit and ignores important contributions to the loss of fringe visibility.

For a spinning astrometric instrument in the HIPPARCOS tradition [e.g., Perryman *et al.*, 1989], there is strong benefit from having multiple view directions. (For a non-spinning high-precision instrument such as Newcomb, OSI, or POINTS, having multiple view directions is essential. However, the use of a precision gyroscope and single view direction has been considered. [Gilmore, CSDL, private communication, ca. 1980, updated 1996]) In HIPPARCOS, the two view directions were provided by a "complex mirror" in front of the telescope primary. This device split the aperture in half along a line in the spin plane, i.e., in the sensitive direction.

We have considered a less equal division of the aperture of length S and "height" H: a central region of length γS would be devoted to one field, and the two remaining regions, each of length $(1-\gamma)S/2$, would be devoted to the other field. All mirror segments would be of full height, as shown in Fig. 1. In an extension of this approach, the central region would contain a complex mirror to create a total of three look directions. I suspect that, to enhance the rigidity of the solution obtained by the analysis of data from a single rotation of the instrument, the look directions of this complex mirror should not be symmetrically placed with respect to the principal look direction.

One naturally asks whether there is a higher information rate in one of these schemes or the other. In this memorandum, I show that as γ increases: (1) the information rate of the principal look direction decreases monotonically; and (2) for the principal region plus the central region, the sum of the information rates remains constant. Note that this is not true of all splittings of the aperture. If one were to split the aperture in half the other way (i.e., along a line

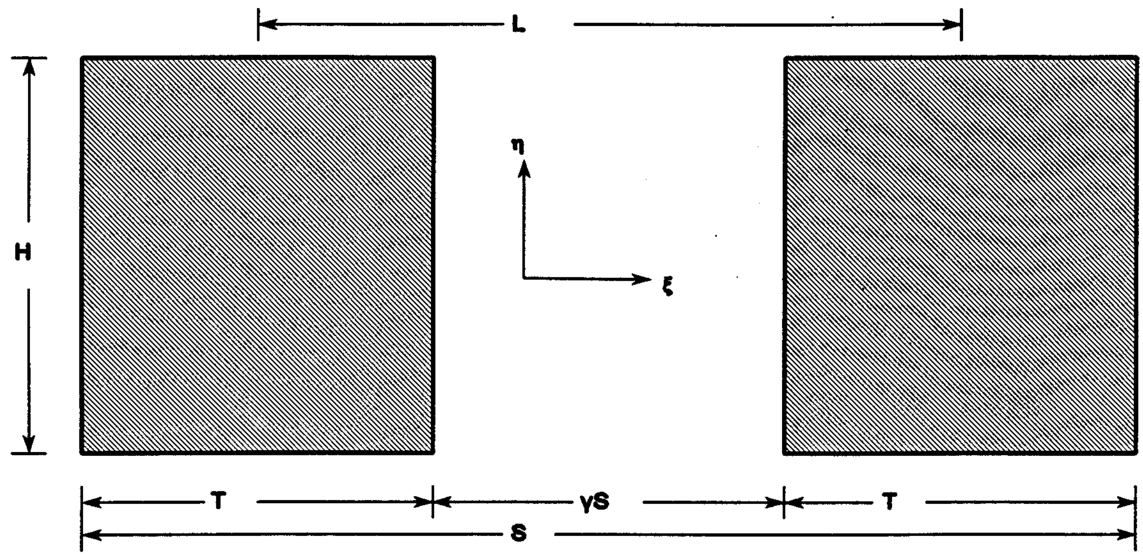

Figure 1. W, the aperture or windowing function, is shown shaded.

perpendicular to the spin plane) and use each half for a separate look direction, the information rate (from the combined look directions) would be decreased by a factor of four.

In Section II, we consider the diffraction pattern of the aperture shown in Fig. 1, and obtain the well-known result for a rectangular aperture and its extension to the spit aperture. In Section III, these results are used to calculate the uncertainty in the measured angular position of a bright target as seen through the aperture of Fig. 1. This result is related to the classical result for a Michelson interferometer.

II. Diffraction Analysis.

The starting point for the analysis is the Fraunhofer integral [Born and Wolf, 1980, p. 386, Eq 43]

$$U(p,q) = \frac{\sqrt{F}}{\lambda} \iint_W e^{-ik(p\xi + q\eta)}\, d\xi\, d\eta \qquad \textbf{(1)}$$

where F is the optical flux (power per unit area), λ is the wavelength, $k = 2\pi/\lambda$, p and q are the image-space angles with respect to the incoming direction (taken to be approximately perpendicular to the aperture plane), ξ and η are coordinates in the aperture plane, and W is the windowing function, i.e., the pupil. U(p,q) is the field, with the (complex) time-dependent part removed, and has units of root power density. We apply the above to the aperture of Fig. 1 and note that, for a rectangular aperture, the problem separates in Cartesian coordinates. There are two directions that will be designated i and c for the interferometrically sensitive and cross direction, respectively. Then

$$U(p,q) = \frac{\sqrt{F}}{\lambda} J_c J_i \tag{2}$$

where for the aperture defined in Fig. 1

$$J_c = \int_{-H/2}^{H/2} e^{-ikq\eta} d\eta = \frac{\sin(\pi q H/\lambda)}{\pi q/\lambda} \tag{3}$$

$$J_i = \int_{-S/2}^{S/2} e^{-ikp\xi} d\xi - \int_{-\gamma S/2}^{\gamma S/2} e^{-ikp\xi} d\xi = \frac{\sin(\pi p S/\lambda) - \sin(\pi p \gamma S/\lambda)}{\pi p/\lambda} \tag{4}$$

(Note that these two are real because the limits are symmetric.) By combining these, we obtain

$$\begin{aligned} U(p,q) &= \frac{\sqrt{F}\,\lambda}{\pi^2 qp} \sin(\pi q H/\lambda)[\sin(\pi p S/\lambda) - \sin(\pi p \gamma S/\lambda)] \\ &= \frac{\sqrt{F}\,HS}{\lambda} \operatorname{sinc}(\pi q H/\lambda)[\operatorname{sinc}(\pi p S/\lambda) - \gamma \operatorname{sinc}(\pi q \gamma S/\lambda)] \end{aligned} \tag{5}$$

where $\operatorname{sinc}(x) \equiv \sin(x)/x$. Finally, as a check on the above, we may calculate the total power in the image plane, which should equal the power passing through the aperture, i.e., the power density times aperture area: $I_a = FHS(1-\gamma)$.

$$I = \iint_{-\infty}^{\infty} [U(p,q)]^2 \, dp \, dq = \frac{F}{\lambda^2} J_q J_p \tag{6}$$

where

$$J_q = \int_{-\infty}^{\infty} J_c^2 dq = \int_{-\infty}^{\infty} \left[\frac{\sin(\pi q H/\lambda)}{\pi q/\lambda}\right]^2 dq = H\lambda \tag{7}$$

$$J_p = \int_{-\infty}^{\infty} J_i^2 \, dp = \int_{-\infty}^{\infty} \left[\frac{\sin(\pi p S/\lambda) - \sin(\pi p \gamma S/\lambda)}{\pi p/\lambda} \right]^2 dp = S\lambda(1-\gamma) \qquad \textbf{(8)}$$

were easily evaluated with the help of Mathcad,®[1] proving the conjecture. In the above integrals, the limits are infinite and the variables of integration are angles. This is a mathematical fiction, physically justified because the range of angles that have significant light is quite small compared to a radian. However, at the level of approximation applied in Born and Wolf to derive Eq 1, one cannot distinguish between p [or q] and tan(p) [or tan(q)]. In the latter case, there would be no problem with the infinite limits of integration.

III. Information rate.

Here we consider the statistical astrometric uncertainty of a star's (angular) position as a function of the aperture parameters S, H, and γ. We ignore sky background, detector read noise and dark current, and loss of fringe visibility from detector discretization, image motion, and finite optical bandwidth. These factors will be the subjects of future analyses.

We need to calculate $\sigma(\hat{p})$, the uncertainty due to photon counting statistics of the estimate of the angular position of an observed star. The "observable" is taken to be the number of photons detected, in an integration time τ, in each pixel of a 2-D detector array. (In practice, only the pixels near the center of the spot would be recorded, but we ignore this restriction here. Further, there would not be a photon count. The observable would be the charge, corresponding to the photon count, collected by the CCD and converted to a digital quantity. This distinction is also ignored.) For now, we assume that an analysis of the data by the method of weighted least squares yields a minimum variance estimate, $\hat{p}$. Then, the covariance matrix P will be the inverse of the coefficient matrix B and the standard deviation of the estimate will be

$$\sigma(p) = \sqrt{P_{pp}} \qquad \textbf{(9)}$$

where the quantity under the radical is the diagonal element of P corresponding to the estimated position $\hat{p}$ in the p direction. We pass to the limit of zero pixel size, and replace sums by integrals. The photon count per cell is replaced by M, a photon-count density,

$$M(p,q) = \frac{\tau}{h\nu} \frac{d^2 I}{dp\, dq} = \frac{\tau U^2}{h\nu} \qquad \textbf{(10)}$$

where h is Plank's constant, $\nu = c/\lambda$ is the optical frequency, and c is the speed of light. Then, the single-parameter coefficient matrix is given by

$$B = \int\!\!\int_{-\infty}^{\infty} \left[\frac{\partial M(p,q)}{\partial p} \frac{1}{\sigma(M)} \right]^2 dp\, dq \tag{11}$$

where $\sigma^2(M) = M$ for Poisson statistics, which is applicable to the photon counts. This expression simplifies to

$$B = \frac{4\tau}{h\nu} \int_{-\infty}^{\infty} dq \int_{-\infty}^{\infty} \left[\frac{\partial U}{\partial p} \right]^2 dp = \frac{4\tau}{h\nu} \frac{F}{\lambda^2} \int_{-\infty}^{\infty} J_c^2\, dq \int_{-\infty}^{\infty} \left[\frac{\partial J_i}{\partial p} \right]^2 dp \tag{12}$$

Here the first of the two integrals is given in Eq 7 and it can be shown that

$$\int_{-\infty}^{\infty} \left[\frac{\partial J_i}{\partial p} \right]^2 dp = \frac{\pi^2 S^3}{3\lambda}(1-\gamma^3) \tag{13}$$

By combining the above results, we find

$$B = \frac{4\pi^2\tau}{3h\nu} \frac{FHS^3}{\lambda^2}(1-\gamma^3) \tag{14}$$

To connect the above to previous results in Michelson interferometry, we note that the number of photons detected is $N = FSH(1-\gamma)\tau/h\nu$, and the baseline length (distance between centers of the subapertures) is $L = S(1+\gamma)/2$. Then

$$B = \left(\frac{2\pi L}{\lambda} \right)^2 N \frac{1+\gamma+\gamma^2}{3} \frac{4}{1+2\gamma+\gamma^2} \tag{15}$$

and

$$\sigma(\hat{p}) = \frac{\lambda}{2\pi L\sqrt{N}} \frac{1}{\sqrt{R}} \tag{16}$$

where

$$R = 1 + \sum_{n=2}^{\infty} \frac{(1-\gamma)^n (n-1)}{3\ 2^n}$$

$$\frac{1}{\sqrt{R}} \approx 1 - \frac{(1-\gamma)^2}{24} - \frac{(1-\gamma)^3}{24} \qquad \textbf{(17)}$$

In the limit of small subapertures, $R \rightarrow 1$, and Eq 16 becomes the standard expression for the astrometric accuracy of a Michelson interferometer. [For a derivation of that equation, see Reasenberg *et al.*, 1996, especially Appendix A, Eq 20.] An alternative approach is to define the baseline length as the RMS separation of the subapertures.

IV. References.

Born, M. and E. Wolf, *Principles of Optics*, Pergamon Press, New York, 1980.

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V. Notes

1. The reference to a commercial product is for technical communication only, and does not constitute an endorsement of the product.

VI. Distribution

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